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Perturbation of Multivariable Linear Quadratic Systems with Jump Parameters and Hybrid Controls

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Abstract: We consider the problem of the perturbation of a class of linear-quadratic systems where the change from one structure (for the dynamics and costs) to another are governed by a finite-state Markov process where the transition are perturbed. The problem above lead to the analysis of some perturbed linearly coupled set of quadratic equations (Riccati Equation). We show that the matrix obtained as the solution of the equations, which determine the optimal value and control, has Taylor expansion in the perturbation parameter. We compute explicitly the term of this expansion.

Key-words: Singular perturbation, stochastic systems, continuous-time Markov chain, small parameter, averaging, Puiseux series and infinite horizon

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Systèmes linéaires quadratiques multidimensionnels avec sauts rapides

Résumé : Nous présentons une étude asymptotique d'un problème de perturbation pour des systèmes linéaires quadratiques multidimensionnels dont la dynamique est modélisée par une équation différentielle. Cette équation dépend, d'une part, de l'état et du contrôle instantané, d'autre part, d'un paramètre pouvant prendre un nombre fini de valeurs. Ce paramètre varie dans le temps selon une dynamique markovienne à saut. Cette partie markovienne de saut évolue beaucoup plus rapidement que la partie continue de l'état du système. Plus précisément, l'intensité stochastique du processus de Markov est multipliée par un facteur $1/\epsilon$.

L'objectif est d'étudier le comportement du système en fonction du paramètre ϵ. En particulier, nous montrons que la valeur optimale et la politique optimale qui contôle la partie continue, possèdent un développement en série de Taylor en le paramètre ϵ; de plus, nous proposons des méthodes numériques efficaces pour le calcul des coefficients de cette série de Taylor.

Mots-clés : Perturbation singulier, Processus Markovien, Série de Puiseux, Agrégation et Horizon infini.

1 Introduction

Many systems subject to frequent unpredictable structural changes can be adequately modelled as piecewise deterministic systems, where the system dynamics takes on different forms depending on the value of an associated Markov process, which is known as form or indicant process associated with the controlled systems. In the linear case, these systems are also known as jump linear system. Such a system model is useful particularly since it allows the decision maker to cope adequately with the discrete events that disrupt and/or change significantly the normal operation of a system by using the knowledge of their occurrence and the statistical information on the rate at which these events take place. Research in this class of systems and their applications into manufacturing management spans several decades, with some representative papers in this area [1, 2, 3, 9, 11, 15].

The traditional **QL** problem in which the system matrices A and B are fixed, we allow these matrices, depending on a Markov jump process with finite-state space. The corresponding dynamic programming equation leads to analysis of a set of Riccati equations involving the generator of the underlying Markov chain. In many applications, the state space of the Markov chain is often large. In this case, it is difficult to obtain solutions to these Riccati equations. To overcome this difficulty, we use singular perturbation techniques in the modelling, control design, and optimization. The resulting systems naturally display certain two-time-scale behavior, a fast time scale and a slowly varying one. Presence of such a phenomenon is best expressed mathematically by introducing a small parameter $\epsilon > 0$ and model the underlying system as one involving singularly perturbed Markov chain.

The introduction of the small parameter stems from the following two aspect. First, in reality, we often have to treat systems that involve rapid jumps leading to the use of two-time-scale. Second, to reduce the complexity of large-scale Markov chain, it is natural to use a small parameter to reflect the different rates of changes among different states leading to singularly perturbed Markovian models with weak and strong interaction. A number of examples given in Yin and Zhang [26]

In this paper we consider the optimal control of a singularly perturbed class of linear-quadratic differential system with piecewise deterministic dynamics. The transition probabilities are perturbed through small parameter $\epsilon > 0$. The corresponding perturbed dynamic programming equation leads to the analysis of a set of perturbed coupled algebraic Riccati equations. By using averaging and aggregation techniques, we show that the set of coupled algebraic Riccati equations possesses a unique solution which can be represented as a Taylor series in ϵ and moreover we present an algorithm for computing the terms of the Taylor series.

Research on control systems with Markovian switching structure was initiated more than thirty years ago by Krassovskii and Liskii [12, 14], and Florentin [10], with follow-up work in the late sixties and early seventies addressing the stochastic maximum principle [13, 19], dynamic programming [19], and linear-quadratic control [20, 24], in this context. Late eighties and early nineties have witnessed renewed interest in the topic, with concentrated research on theoretical issues like controllability and stabilizability [4, 9, 11, 22, 25] in linear-quadratic systems in continuous time, see also [5, 18, 21] for discrete-time. Perhaps the first theoretical development in differential games context was reported in [3], where a general model was adopted that allows in a multiple player environment the Markov jump process (also controlled by the players) to determine the mode of the game, in addition to affecting the system and cost structure. Results in [17], also apply, as a special case, to zero-sum differential games with Markov jump parameters and state feedback information for both players. Some selected publications on this topic are [2, 22, 25].

An important paper related to ours is [16] which considers also linear dynamics with jump parameters. The transition probabilities are subject to the same type of perturbation that we consider. A more general setting is treated in that reference: that of an additional disturbance, which is handled using an H^∞ approach. The value of the limiting control problem is obtained. Our approach allows us to get more than the limiting control problem, as it gives the solutions in terms of a series expansion in the perturbation parameter. Our paper also extends the results in [8] which already obtained a Taylor expansion for the optimal value and policy in a similar setting, but for the special case of a one dimensional linear system with a controlled Markov jump process.

The paper is organized as follows: Section 2 introduces the general model. In section 3, we present the motivation for singularly perturbed problems. Section 4 provides the Taylor expansion of the solution to the coupled algebraic Riccati equations. We study computational algorithms in section 5. The paper ends with the concluding remarks of section 6.

2 General Model

The class of jump linear systems under consideration is described by

$$\frac{dx}{dt} = A(\theta(t))x(t) + B(\theta(t))u(t), \qquad x(0) = x_0, \tag{1}$$

where x is the p-dimensional system state, x_0 is a fixed (known) initial state, u is an r-dimensional control, taking values in $\mathbb{R}^r$, and $\theta(t)$ is a finite state Markov chain defined on the state space S, of cardinality s, with the infinitesimal generator matrix

$$\Lambda = (\lambda_{ij}), \qquad i, j \in S.$$

The λ_{ij}'s are real numbers such that for any $i \neq j$, $\lambda_{ij} \geq 0$, and for all $i \in S$, $\lambda_{iai} = -\sum_{j \neq i} \lambda_{iaj}$.
With this system, the control u is generated by a control policy γ according to

$$u(t) = \gamma(t, x_{[0,t]}, \theta_{[0,t]}), \qquad t \in [0, +\infty), \tag{2}$$

where γ is taken to be piecewise continuous in its first argument, and piecewise Lipschitz continuous in its second argument. Let us denote the class of all admissible controllers by Γ. Define the running (immediate) cost $L : \mathbb{R}^p \times S \times \Gamma \to \mathbb{R}$ as:

$$L(x, i, u) = \mid x \mid^2_{Q(i)} + \mid u \mid^2_{R(i)}, \tag{3}$$

where $Q(.) \geq 0$, $R(.) > 0$, and $\mid x \mid_Q$ denotes the Euclidean semi-norm.
The underlying probability space is the triple (Ω, F, P). Let E denote the expectation with respect to the underlying probability space. For each initial state (x_0, i_0) and strategy $\gamma \in \Gamma$, we introduce the discounted (expected) cost function:

$$J_\beta(x_0, i_0, \gamma) = E_{x_0, i_0} \left(\int_0^{+\infty} e^{-\beta t} L(x(t), \theta(t), u(t)) dt \right), \tag{4}$$

where $\beta \geq 0$ is a discount factor.

For this infinite-horizon, we have to ensure that the cost is finite for at least one stationary policy. A sufficient condition for this is the following (see also Remark 2):

Assumption 1:

The pair $(A(\theta(t)), B(\theta(t)))$ is stochastically stabilizable and $(A(i), Q(i))$ is observable for each $i \in S$.

The problem is the derivation of a solution to

$$\hat{J}_\beta(x, i) = \inf_{\gamma \in \Gamma} J_\beta(x, i, \gamma). \tag{5}$$

3 Singularly perturbed systems.

In many application, because of the various sources of uncertainties, the Markov chain involved is often inevitably large dimensional. It is natural ti think of the large number of states to be grouped into different collections of states, based on whether the interaction between any two states is weak or strong. Presence of such a phenomenon is best expressed mathematically by taking the probability transition rate matrix Λ in an appropriate singularly perturbed form, as already discussed in [6] and [7]:

$$\lambda_{ij}^{\epsilon} = \nu_{ij} + \frac{1}{\epsilon}\mu_{ij}, \tag{6}$$

where $(\nu_{ij})_{s\times s}$ and $(\mu_{ij})_{s\times s}$ are the transition probability rate matrices corresponding to, respectively, weak interaction and strong interactions within the form process. The scalar ϵ is a small positive number.

We study the optimal **LQ** control problem by using the dynamic programming. Under assumption 1, the optimal value $\hat{J}_{\beta}^{\epsilon}$ satisfies the Hamilton-Jacobi-Bellman equations (Theorem 5 [11]) and this solution is given by the quadratic form $x^T P_{\epsilon}(i)x$ where $P_{\epsilon}(i)$ is a $p\times p$ symmetric matrix for each $i \in S$ which is the unique positive definite solution [11] of the following perturbed coupled algebraic matrix linearly Riccati equations:

$$\begin{aligned} \beta P_{\epsilon}(i) &= A^T(i)P_{\epsilon}(i) + P_{\epsilon}(i)A(i) + Q(i) - P_{\epsilon}(i)B(i)R^{-1}(i)B^T(i)P_{\epsilon}(i) \\ &+ \sum_{j\in S}\lambda_{ij}^{\epsilon}P_{\epsilon}(j), \quad i \in S. \end{aligned} \tag{7}$$

Moreover, the optimal control is given by

$$\gamma_{opt}^{\epsilon}(x,i) = -R^{-1}(i)B^T(i)P_{\epsilon}(i)x.$$

For any fixed ϵ, one can find the optimal value and an optimal solution by solving the Riccati equation. The advantage of presenting a Taylor expansion in ϵ is to avoid having to solve a Riccati equation for each small ϵ. This expansion will be shown to have the useful feature that the number of coupled Riccati equations involved in determining the coefficients is smaller than in the original one. Thus the optimal value $\hat{J}_{\beta}^{\epsilon}$ and the optimal controller are determined for small value of $\epsilon > 0$ by solving well-behaved ϵ-independent smaller problems.

4 The Taylor expansion.

In this section, we study the problem formulated in the previous section under the following assumption which is stronger than Assumption 1.

Assumption 2

The matrices $B(i)$ are nonsingular for any $i \in S$ and $(A(i), Q(i))$ is observable for each $i \in S$.

Precisely the problem addressed in this section is to determine explicitly the solution of Riccati equation according to the small parameter ϵ.

We consider the Markov chain associated with the transition probability rates $\mu = (\mu_{ij})_{i,j \in S}$. There exists a partition of S into a family of c recurrent classes $\bar{\xi}_1, \bar{\xi}_2, .., \bar{\xi}_c$, and a transient class T:

$$S = (\cup_{n=1}^{c} \bar{\xi}_n) \cup T \quad with \quad \bar{\xi}_n \cap \bar{\xi}_{n'} = \emptyset \quad if \quad n \neq n'.$$

Hence

$$\mu_{ij} = 0 \quad if \quad i \in \bar{\xi}_n \quad and \quad j \in \bar{\xi}_{n'}, \quad n \neq n'.$$

To each class $\bar{\xi}$ is associated the invariant measure (row vector) $m_{\bar{\xi}}$ of the recurrent subchain defined on the class $\bar{\xi} \in \bar{S}$ where $\bar{S} = \{\bar{\xi}_1, \bar{\xi}_2, .., \bar{\xi}_c\}$. We shall denote by $q_{\bar{\xi}}(i)$ the probability to end in the class $\bar{\xi}$ starting from i. The c functions (column vectors) $q_{\bar{\xi}}(.)$ are the solutions to $\mu q_{\bar{\xi}} = 0$, and form a basis of the c-dimensional subspace $\mathrm{Ker}(\mu)$.

Remark 1 *If v is a solution to $\sum_{j \in S} \mu_{ij} v(j) = 0$, then*

$$\begin{aligned} v(i) &= v(\bar{\xi}), \qquad \forall i \in \bar{\xi} \\ v(i) &= \sum_{\bar{\xi} \in \bar{S}} q_{\bar{\xi}}(i) v(\bar{\xi}), \qquad i \in T, \end{aligned}$$

where $(v(\bar{\xi}))_{\bar{\xi}}$ are some real numbers.

Theorem 4.1
The solution of equation (7) is a Puiseux series of the following form. There exists a positive integer M such that

$$P_\epsilon(i) = \sum_{n=-M}^{+\infty} P_n(i) \epsilon^{n/M},$$

where $P_n(i)$ is $p \times p$ symmetric matrix for each $i \in S$.

Proof
From Puiseux's Theorem [23], we have

$$P_\epsilon(i) = \sum_{n=-K}^{+\infty} P_n(i) \epsilon^{n/M} \tag{8}$$

for some integers K and M. Substituting the expression of (8) in (7), we obtain:

$$\begin{aligned}
\beta \sum_{n=-K}^{+\infty} P_n(i)\epsilon^{n/M} &= Q(i) + A^T(i) \sum_{n=-K}^{+\infty} P_n(i)\epsilon^{n/M} + \sum_{n=-K}^{+\infty} P_n(i)\epsilon^{n/M} A(i) && (9)\\
&- \sum_{n=-2K}^{+\infty} \sum_{k=-2K}^{n} \left(P_{k+K}(i)B(i)R^{-1}(i)B^T(i)P_{n-(k+K)}(i)\right)\epsilon^{n/M} && (10)\\
&+ \sum_{n=-K}^{+\infty} \sum_{j\in S} \nu_{ij}P_n(j)\epsilon^{n/M} + \sum_{n=-K-M}^{+\infty} \sum_{j\in S} \mu_{ij}P_{n+M}(j)\epsilon^{n/M}, && (11)
\end{aligned}$$

Next, we equate the coefficients of $\epsilon^{n/M}$ for different n's. We have to show that $K \leq M$. Assume in the contrary that $K > M$. We shall show that all coefficients that correspond to $n < -M$ are zero. First set $n = -2K$ in the above equation. Then $P_n(i) = 0$ for all $i \in S$. Moreover, $-2K < -M - K$, so that there are no coefficients of $\epsilon^{n/M}$ except for those appearing in (10). In (10) the number of terms in the second summation is one, which gives for any $i \in S$

$$P_{-K}(i)B(i)R^{-1}(i)B^T(i)P_{-K}(i) = 0. \tag{12}$$

Since $R^{-1}(i) > 0$ and $B(i)$ is nonsingular, then $P_{-K}(i) = 0$.
If $n = -2K + 1 < -M - K$, then by the same arguments we get

$$P_{-K+1}(i)B(i)R^{-1}(i)B^T(i)P_{-K+1}(i) = 0, \;\; i \in S.$$

Since $R^{-1}(i) > 0$ and $B(i)$ is nonsingular, then $P_{-K+1}(i) = 0$.
If we continue with same procedure for $-2K \leq n < -K - M$, it follows that:

$$P_{-K}(i) = P_{-K+1}(i)... = P_{-M-1}(i) = 0, \;\; i \in S,$$

and this concludes the proof. ∎

The previous theorem showed that the value and solution of the control problem have a fractional expansion in ϵ. Next we shall show that in fact, all coefficients corresponding to non-integer powers of ϵ vanish, and that we obtain a Taylor series.

From Theorem 4.1 and (7) it follows that:

$$\begin{aligned}
\beta \sum_{n=-M}^{+\infty} P_n(i)\epsilon^{n/M} &= Q(i) + A^T(i) \sum_{n=-M}^{+\infty} P_n(i)\epsilon^{n/M} + \sum_{n=-M}^{+\infty} P_n(i)\epsilon^{n/M} A(i)\\
&- \sum_{n=-2M}^{+\infty} \sum_{k=-2M}^{n} \left(P_{k+M}(i)B(i)R^{-1}(i)B^T(i)P_{n-(k+M)}(i)\right)\epsilon^{n/M}
\end{aligned}$$

$$+ \sum_{n=-M}^{+\infty} \sum_{j \in S} \nu_{ij} P_n(j) \epsilon^{n/M} + \sum_{n=-2M}^{+\infty} \sum_{j \in S} \mu_{ij} P_{n+M}(j) \epsilon^{n/M},$$

Then we obtain the following set of equations:
If $-2M \leq n < -M$, then

$$0 = -\sum_{k=-2M}^{n} P_{k+M}(i) B(i) R^{-1}(i) B^T(i) P_{n-(k+M)}(i) + \sum_{j \in S} \mu_{ij} P_{n+M}(j), \quad i \in S.$$

If $-M \leq n < 0$, then

$$\begin{aligned} \beta P_n(i) &= A^T(i) P_n(i) + P_n(i) A(i) - \sum_{k=-2M}^{n} P_{M+k}(i) B(i) R^{-1}(i) B^T(i) P_{n-(k+M)}(i) \\ &+ \sum_{j \in S} \nu_{ij} P_n(j) + \sum_{j \in S} \mu_{ij} P_{n+M}(j), \quad i \in S. \end{aligned}$$

If $n = 0$, then

$$\begin{aligned} &\beta P_0(i) \\ &= Q(i) + A^T(i) P_0(i) + P_0(i) A(i) - \sum_{k=-2M}^{0} P_{M+k}(i) B(i) R^{-1}(i) B^T(i) P_{-(k+M)}(i) \\ &\quad + \sum_{j \in S} \nu_{ij} P_0(j) + \sum_{j \in S} \mu_{ij} P_M(j), \quad i \in S. \end{aligned}$$

If $n > 0$, then

$$\begin{aligned} \beta P_n(i) &= A^T(i) P_n(i) + P_n(i) A(i) - \sum_{k=-2M}^{n} P_{M+k}(i) B(i) R^{-1}(i) B^T(i) P_{n-(k+M)}(i) \\ &+ \sum_{j \in S} \nu_{ij} P_n(j) + \sum_{j \in S} \mu_{ij} P_{n+M}(j), \quad i \in S. \end{aligned}$$

From the above set of equations, it follows that, if $n = -2M$, then

$$\sum_{j \in S} \mu_{ij} P_{-M}(j) = P_{-M}(i) B(i) R^{-1}(i) B^T(i) P_{-M}(i), \quad i \in S. \tag{13}$$

Since the equation (7) admits a unique positive definite solution for ϵ small enough, then $\forall i \in S$, $\forall x \in \mathbb{R}^p$, $x^T P_{-M}(i) x \geq 0$. Fix x arbitrarily in $\mathbb{R}^p$ and let $i^* = \arg\max_{i \in S} x^T P_{-M}(i) x$. Then

$$\Big(\sum_{\substack{j \in S \\ j \neq i^*}} \mu_{i^* j}\Big) x^T [P_{-M}(i^*)] x \geq \sum_{\substack{j \in S \\ j \neq i^*}} \mu_{i^* j} x^T P_{-M}(j) x$$

and it follows that:

$$\sum_{j\in S}\mu_{i^*j}x^TP_{-M}(j)x \le 0.$$

From (13), since $R^{-1}(i) > 0$ and $B(i)$ is nonsingular, then $x^TP_{-M}(i^*)x = 0$ and $x^TP_{-M}(i)x = 0$ $\forall i \in S$, $\forall x \in \mathbb{R}^p$, hence $P_{-M}(i) = 0$ for all $i \in S$.
If $n = -2M + 1 < -M$, we get (since $P_{-M}(i) = 0$),

$$\sum_{j\in S}\mu_{ij}P_{-M+1}(j) = 0. \tag{14}$$

If $n = -2M + 2 < -M$, then

$$\sum_{j\in S}\mu_{ij}P_{-M+2}(j) = P_{-M+1}(i)B(i)R^{-1}(i)B^T(i)P_{-M+1}(i) \tag{15}$$

From (14) and Remark 1, we obtain:

$$\bar{P}_{-M+1}(\bar{\xi}) := P_{-M+1}(i) \text{ for all } i \in \bar{\xi}$$

and

$$P_{-M+1}(i) = \sum_{\bar{\xi}\in\bar{S}} q_{\bar{\xi}}(i)\bar{P}_{-M+1}(\bar{\xi}) \text{ for all } i \in T.$$

Now, equation (15) can be written as:

$$\sum_{j\in S}\mu_{ij}P_{-M+2}(j) = \bar{P}_{-M+1}(\bar{\xi})B(i)R^{-1}(i)B^T(i)\bar{P}_{-M+1}(\bar{\xi}), \quad i \in \bar{\xi}.$$

Multiplying the last equation by $m_{\bar{\xi}}(i)$ for each $i \in \bar{\xi}$ and summing over $\bar{\xi}$, we obtain:

$$\sum_{j\in S}\left(\underbrace{\sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)\mu_{ij}}_{=0}\right)P_{-M+2}(j) = \bar{P}_{-M+1}(\bar{\xi})\left(\sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)[B(i)R^{-1}(i)B^T(i)]\right)\bar{P}_{-M+1}(\bar{\xi}).$$

Since $R^{-1}(i) > 0$ and $B(i)$ is nonsingular, then $P_{-M+1}(i) = 0$, $i \in S$ and $\sum_{j\in S}\mu_{ij}P_{-M+2}(j) = 0$, $i \in S$. By using the same procedure for the case where $-2M \le n < -M$, we obtain:

$$P_{-M}(i) = ... = P_{\lfloor(-M-1)/2\rfloor}(i) = 0 \quad i \in S, \tag{16}$$

where $\lfloor y \rfloor$ stands for the greatest integer which is smaller or equal to y and

$$\sum_{j\in S}\mu_{ij}P_k(j) = 0, \quad k = \lfloor(-M+1)/2)\rfloor, ..., -1. \tag{17}$$

In the sequel, we consider two cases for $-M \leq n < 0$
Case 1: M is even
If $n = -M$, then

$$\sum_{j \in S} \mu_{ij} P_0(j) = P_{\frac{-M}{2}}(i) B(i) R^{-1}(i) B^T(i) P_{\frac{-M}{2}}(i), \qquad i \in S. \tag{18}$$

By using the same technique to derive equations (14) and (15), we obtain

$$P_{-\frac{M}{2}}(i) = 0 \quad and \quad \sum_{j \in S} \mu_{ij} P_0(j) = 0. \tag{19}$$

If $n = -M+1$, then

$$\sum_{j \in S} \mu_{ij} P_1(j) = 0, \; i \in S.$$

If $n = -M+2 < 0$, then

$$\sum_{j \in S} \mu_{ij} P_2(j) = P_{\frac{-M+2}{2}}(i) B(i) R^{-1}(i) B^T(i) P_{\frac{-M+2}{2}}(i), \; i \in S.$$

It follows that

$$P_{\frac{-M+2}{2}}(i) = 0 \;\; , \;\; \sum_{j \in S} \mu_{ij} P_2(j) = 0.$$

If we continue with the same procedure for $-M \leq n < 0$, it follows that:

$$P_{\frac{-M}{2}}(i) = ... = P_{-1}(i) = 0, \quad i \in S, \tag{20}$$

and

$$\sum_{j \in S} \mu_{ij} P_k(j) = 0 \quad for \quad k = 0, 1, ..., M-1. \tag{21}$$

Case 2: M is odd
If $n = -M$, then

$$\sum_{j \in S} \mu_{ij} P_0(j) = 0, \quad i \in S. \tag{22}$$

If $n = -M+1 < 0$, then

$$\sum_{j \in S} \mu_{ij} P_1(j) = P_{\frac{-M+1}{2}}(i) B(i) R^{-1}(i) B^T(i) P_{\frac{-M+1}{2}}(i), \;\; i \in S.$$

Therefore

$$P_{\frac{-M+1}{2}}(i) = 0 \quad , \quad \sum_{j \in S} \mu_{ij} P_1(j) = 0, \; i \in S.$$

By using the same procedure as in the case where M is even, we obtain analogous results.

If $n = 0$, we get:

$$\begin{aligned}\beta P_0(i) &= Q(i) + A^T(i)P_0(i) + P_0(i)A(i) - P_0(i)B(i)R^{-1}(i)B^T(i)P_0(i) + \sum_{j\in S}\nu_{ij}P_0(j) \\ &\quad + \sum_{j\in S}\mu_{ij}P_M(j), \quad i \in S. \end{aligned} \tag{23}$$

From (21) or (22), we have that:

$$\sum_{j\in S}\mu_{ij}P_0(j) = 0, \quad i \in S. \tag{24}$$

Using the equation (24) and Remark 1, we obtain

$$\begin{cases} \bar{P}_0(\bar{\xi}) = P_0(i), & i \in \bar{\xi} \\ P_0(i) = \sum_{\bar{\xi}\in\bar{S}} q_{\bar{\xi}}(i)\bar{P}_0(\bar{\xi}), & i \in T. \end{cases} \tag{25}$$

Substituting the first expression of (25) in (23) to get:

$$\beta\bar{P}_0(\bar{\xi}) = Q(i) - \bar{P}_0(\bar{\xi})B(i)R^{-1}(i)B^T(i)\bar{P}_0(\bar{\xi}) + \sum_{j\in S}\mu_{ij}P_M(j) + \sum_{j\in S}\nu_{ij}P_0(j), \quad i \in \bar{\xi}. \tag{26}$$

Multiplying (26) by $m_{\bar{\xi}}(i)$, $i \in \bar{\xi}$ and summing up over $\bar{\xi}$:

$$\begin{aligned}\sum_{i\in\bar{\xi}}\left[m_{\bar{\xi}}(i)\beta\right]\bar{P}_0(\bar{\xi}) &= \sum_{i\in\bar{\xi}}Q(i)m_{\bar{\xi}}(i) + \sum_{i\in\bar{\xi}}m_{\bar{\xi}}(i)A^T(i)\bar{P}_0(\bar{\xi}) + \bar{P}_0(\bar{\xi})\sum_{i\in\bar{\xi}}m_{\bar{\xi}}(i)A(i) \\ &- \bar{P}_0(\xi)\sum_{i\in\bar{\xi}}\left[m_{\bar{\xi}}(i)B(i)R^{-1}(i)B^T\right]\bar{P}_0(\bar{\xi}) + \sum_{j\in S}\left(\underbrace{\sum_{i\in\bar{\xi}}m_{\bar{\xi}}(i)\mu_{ij}}_{=0}\right)P_M(j) \\ &+ \sum_{\bar{\xi}'\in\bar{S}}\left(\sum_{\substack{i\in\bar{\xi}\\ j\in\bar{\xi}'}}m_{\bar{\xi}}(i)\nu_{ij}\right)\bar{P}_0(\bar{\xi}') + \sum_{j\in T}\sum_{i\in\bar{\xi}}\sum_{\bar{\xi}'\in\bar{S}}m_{\bar{\xi}}(i)\nu_{ij}q_{\bar{\xi}'}(j)\bar{P}_0(\bar{\xi}'). \end{aligned}\tag{27}$$

We introduce the following notations:

$$\bar{A}(\bar{\xi}) = \sum_{i\in\bar{\xi}}m_{\bar{\xi}}(i)A(i), \qquad \bar{Q}(\bar{\xi}) = \sum_{i\in\bar{\xi}}m_{\bar{\xi}}(i)Q(i),$$

$$\bar{B}(\bar{\xi}) = \left(\sum_{i \in \bar{\xi}} m_{\bar{\xi}}(i) B(i) R^{-1}(i) B^T(i) \right)^{1/2}$$

and

$$\nu_{\bar{\xi}\bar{\xi}'} = \sum_{i \in \bar{\xi}} \left(\sum_{j \in \bar{\xi}'} \nu_{ij} + \sum_{j \in T} q_{\bar{\xi}'}(j) \nu_{ij} \right).$$

Thus, the equation (27) becomes:

$$\begin{aligned} \beta \bar{P}_0(\bar{\xi}) &= \bar{Q}(\bar{\xi}) + \bar{A}^T(\bar{\xi})\bar{P}_0(\bar{\xi}) + \bar{P}_0(\bar{\xi})\bar{A}(\bar{\xi}) - \bar{P}_0(\bar{\xi})\bar{B}(\bar{\xi})\bar{B}^T(\bar{\xi})\bar{P}_0(\bar{\xi}) \\ &+ \sum_{\bar{\xi}' \in \bar{S}} \nu_{\bar{\xi}\bar{\xi}'} \bar{P}_0(\bar{\xi}'), \quad \bar{\xi} \in \bar{S}. \end{aligned} \tag{28}$$

Note that $\bar{\theta}(t)$ defines an aggregated Markov chain defined on the state space $\bar{S}$ with the infinitesimal generator matrix $\bar{\Gamma} = (\nu_{\bar{\xi}\bar{\xi}'})_{c \times c}$

Assumption 3

The pair $(\bar{A}(\bar{\theta}), \bar{B}(\bar{\theta}))$ is stochastically stabilizable and $(\bar{A}(\bar{\xi}), \bar{Q}(\bar{\xi}))$ is observable for each $\bar{\xi} \in \bar{S}$.

Under Condition 2, the equation (28) admits a unique positive solution, hence P_0 is uniquely determined by (23) and (24). Furthermore, the following jump linear system is mean-square stable [11]:

$$\dot{x} = D(\bar{\theta})x, \tag{29}$$

where $D(\bar{\xi}) = \bar{A}(\bar{\xi}) - \bar{B}(\bar{\xi})\bar{B}^T(\bar{\xi})\bar{P}_0(\bar{\xi})$.
If $n = 1$, we have

$$\begin{aligned} \beta P_1(i) &= A^T(i)P_1(i) + P_1(i)A(i) - P_0(i)B(i)R^{-1}(i)B^T(i)P_1(i) \\ &- P_1(i)B(i)R^{-1}(i)B^T(i)P_0(i) + \sum_{j \in S} \nu_{ij} P_1(j) + \sum_{j \in S} \mu_{ij} P_{M+1}(j), \quad i \in S, \end{aligned} \tag{30}$$

From (21), we have:

$$\sum_{j \in S} \mu_{ij} P_1(j) = 0. \tag{31}$$

By using (31) and Remark 1, we obtain:

$$\begin{cases} \bar{P}_1(\bar{\xi}) = P_1(i), & i \in \bar{\xi} \\ P_1(i) = \sum_{\bar{\xi} \in \bar{S}} q_{\bar{\xi}}(i) \bar{P}_1(\bar{\xi}), & i \in T. \end{cases} \tag{32}$$

Substituting the first expression of (32) in (30) to get

$$\begin{aligned} \beta\bar{P}_1(\bar{\xi}) &= A^T(i)\bar{P}_1(\bar{\xi}) + \bar{P}_1(\bar{\xi})A(i) - \bar{P}_0(\bar{\xi}))B(i)R^{-1}(i)B^T(i)\bar{P}_1(\bar{\xi}) \\ &- \bar{P}_1(\bar{\xi}))B(i)R^{-1}(i)B^T(i)\bar{P}_0(\bar{\xi}) + \sum_{\bar{\xi}'\in\bar{S}}\sum_{j\in\bar{\xi}'}\nu_{ij}\bar{P}_1(\bar{\xi}') + \sum_{j\in S}\mu_{ij}P_{M+1}(j). \quad (33) \end{aligned}$$

Multiplying (33) by $m_{\bar{\xi}}(i)$, $i \in \bar{\xi}$ and summing over $\bar{\xi}$, we obtain:

$$\beta\bar{P}_1(\bar{\xi}) = D^T(\bar{\xi})\bar{P}_1(\bar{\xi}) + \bar{P}_1(\bar{\xi})D(\bar{\xi}) + \sum_{\bar{\xi}'\in\bar{S}}\nu_{\bar{\xi}\bar{\xi}'}\bar{P}_1(\bar{\xi}'). \quad (34)$$

Since the model (29) is mean-square stable, one applies Proposition 2 [4] to show that equation (34) has a unique solution. Since 0 is solution to (34), then 0 is the unique solution to (34). Thus, it follows from $P_1(i) = 0$, $i \in S$ and (30) that

$$\sum_{j\in S}\mu_{ij}P_{M+1}(j) = 0, \; i \in S. \quad (35)$$

If $M \neq 1$, then for $0 < n < M$, we have

$$\begin{aligned} \beta P_n(i) &= A^T(i)P_n(i) + P_n(i)A(i) - P_0(i)B(i)R^{-1}(i)B^T(i)P_n(i) \\ &- P_n(i)B(i)R^{-1}(i)B^T(i)P_0(i) + \sum_{j\in S}\mu_{ij}P_{n+M}(j) + \sum_{j\in S}\nu_{ij}P_n(j). \end{aligned}$$

From (21), it follows that $\sum_{j\in S}\mu_{ij}P_n(j) = 0$. Then we can derive $P_1(i) = .. = P_{M-1} = 0$, and

$$\sum_{j\in S}\mu_{ij}P_{n+M}(j) = 0 \qquad n = 0, 1, ..., M-1. \quad (36)$$

If $n = M$, then

$$\begin{aligned} \beta P_M(i) &= A^T(i)P_M(i) + P_M(i)A(i) - P_0(i)N(i)P_M(i) - P_M(i)N(i)P_0(i) \\ &+ \sum_{j\in S}\mu_{ij}P_{2M}(j) + \sum_{j\in S}\nu_{ij}P_M(j), \end{aligned}$$

where $N(i) = B(i)R^{-1}(i)B^T(i)$. From (23), we get

$$\sum_{j\in S}\mu_{ij}P_M(j) = \alpha(i), \qquad i \in S, \quad (37)$$

where $\alpha(i) = \beta P_0(i) - A^T(i)P_0(i) - P_0(i)A(i) + P_0(i)N(i)P_0(i) - Q(i) - \sum_{j\in S}\nu_{ij}P_0(j)$. If $M \neq 1$, then for $M < n < 2M$, we have

$$\begin{aligned} \beta P_n(i) &= A^T(i)P_n(i) + P_n(i)A(i) - P_0(i)N(i)P_n(i) - P_n(i)N(i)P_0(i) \\ &+ \sum_{j\in S}\mu_{ij}P_{n+M}(j) + \sum_{j\in S}\nu_{ij}P_n(j). \end{aligned}$$

From(36), we have that $\sum_{j \in S} \mu_{ij} P_n(j) = 0$, and therefore $P_n(j) = 0, \ \forall j \in S \ and \ n = M+1, .., 2M-1$.
Similarly, we have that:

$$\begin{array}{rcl} P_n(i) & = & 0, \quad n < 0 \\ P_0(i) & > & 0, \quad i \in S \\ P_n(i) & = & 0, \quad n \neq kM, \quad k \in \mathbb{N}^*. \end{array}$$

Now, we can state the following result.

Theorem 4.2 *Let assumption 2 and 3 hold. Then the coupled algebraic Riccati equation (7) admits a unique positive solution, which can be expanded as a Taylor series.*

Remark 2 *It follows from the proof of Theorem 1 in [15] (without disturbance) that the jump linear system (1) is stochastically stabilizable for sufficiently small* $\epsilon > 0$*, if the aggregate jump linear system (29) is stochastically stabilizable, and with assumptions 1, 2, 5, and 6 in [15], the solutions of the perturbed coupled algebraic Riccati equations (7) are of the form* $P + O(\epsilon)$.

We introduce the following condition:

Assumption 4:

The initial probability distribution of the Markov chain $\theta(t)$ is positive for any state of teh Markov chain.

A by product of Remark 2, Theorem 4.2 and its proof, is the result given in the following corollary:

Corollaire 1 *Let assumption 3 and 4 hold and assume that the jump linear system (1) is observable. Then there exists* ϵ_0 *such that the perturbed coupled algebraic Riccati equations (7) admits a unique positive solution which can be expanded as a Taylor series, for any* $\epsilon \in (0, \epsilon_0]$.

5 Computation algorithms

The next theorem is about the computation of the terms in the expansion of

P_ϵ and moreover the proof of the theorem is constructive in the sense that it provides an algorithm for calculation.

Notation

Let $Ker(C)$ denote the Kernel and $Im(C)$ the range of the operator $C(f)$, i.e.

$$Ker(C) = \{y \in \mathbb{R}^s / Cy = 0\} \quad \text{and} \quad Im(C) = \{y \in \mathbb{R}^s / \exists x \in \mathbb{R}^s, y = Cx\}.$$

Theorem 5.1 *Let assumption 2 and 3 hold. Then the solution of the algebraic Riccati equation (7) P_ϵ has the expansion $\sum_{n=0}^{+\infty} P_n(i)\epsilon^n$, where $(P_n)_{kl} = (\tilde{P}_n)_{kl} + (\bar{P}_n)_{kl}$ and $(\tilde{P}_n)_{kl} := ((\tilde{P}_n(i))_{kl})_{i\in S} \in Im(\mu)$ and $(\bar{P}_n)_{kl} = ((\bar{P}_n(i))_{kl})_{i\in S} \in Ker(\mu)$. The sequence $((\tilde{P}_n), (\bar{P}_n))$ is uniquely determined by the following iterative algorithm:*

1. *$P_0(i)$, $i \in S$ is given by:*

$$\begin{cases} P_0(i) = \bar{P}_0(\bar{\xi}), & i \in \bar{\xi}, \quad \forall \bar{\xi} \in \bar{S}. \\ P_0(i) = \sum_{\bar{\xi}\in\bar{S}} q_{\bar{\xi}}(i)\bar{P}_0(\bar{\xi}), & i \in T. \end{cases}$$

where $P_0(\bar{\xi})$, $\bar{\xi} \in \bar{S}$ is a unique solution of the Coupled Algebraic Riccati Equations:

$$\beta\bar{P}_0(\bar{\xi}) = \bar{Q}(\bar{\xi}) + A^T(\bar{\xi})\bar{P}_0(\bar{\xi}) + \bar{P}_0(\bar{\xi})A(\bar{\xi}) - \bar{P}_0(\bar{\xi})\bar{B}(\bar{\xi})\bar{B}^T(\bar{\xi})\bar{P}_0(\bar{\xi}) + \sum_{\bar{\xi}'\in\bar{S}} \nu_{\bar{\xi}\bar{\xi}'}\bar{P}_0(\bar{\xi}'), \quad \bar{\xi} \in \bar{S}. \tag{38}$$

2. *$\tilde{P}_n(i)$, $i \in S$, $n > 0$ is a unique solution to the linear system:*

$$\mu(\tilde{P}_n)_{kl} = (\alpha_{n-1})_{kl}, \tag{39}$$

where

$$\begin{aligned} \alpha_0(i) &= \beta P_0(i) - Q(i) - A^T(i)P_0(i) - P_0(i)A(i) + P_0(i)B(i)R^{-1}(i)B^T(i)P_0(i) \\ &\quad - \sum_{j\in S} \nu_{ij}P_0(j), \end{aligned}$$

and

$$\begin{aligned} \alpha_n(i) &= \beta P_n(i) - A^T(i)P_n(i) - P_n(i)A(i) + P_0(i)B(i)R^{-1}(i)B^T(i)P_n(i) \\ &+ P_n(i)B(i)R^{-1}(i)B^T(i)P_0(i) - \sum_{j\in S} \nu_{ij}P_n(j) + f_n(i), \quad n > 0, \end{aligned}$$

with $f_n(i) = \sum_{k=1}^{n-1} P_k(i)B(i)R^{-1}(i)B^T(i)P_{n-k}(i)$.

3. $\bar{P}_n(i)$, $i \in S$, $n > 0$ is given by:

$$\begin{cases} \bar{P}_n(i) = \bar{P}_n(\bar{\xi}), & i \in \bar{\xi}, \quad \forall \bar{\xi} \in \bar{S}. \\ P_n(i) = \sum_{\bar{\xi} \in \bar{S}} q_{\bar{\xi}}(i)\bar{P}_0(\bar{\xi}), & i \in T. \end{cases}$$

where $\bar{P}_n(\bar{\xi})$, $\bar{\xi} \in \bar{S}$ is a unique solution of the following Lyapunov equations:

$$\beta \bar{P}_n(\bar{\xi}) = D^T(\bar{\xi})\bar{P}_n(\bar{\xi}) + \bar{P}_n(\bar{\xi})D(\bar{\xi}) + \sum_{\bar{\xi}' \in \bar{S}} \nu_{\bar{\xi}\bar{\xi}'} \bar{P}_n(\bar{\xi}') + T_n(\bar{\xi}), \quad \bar{\xi} \in \bar{S}, \tag{40}$$

where

$$\begin{aligned} T_n(\bar{\xi}) &= \sum_{i \in \bar{\xi}} m_{\bar{\xi}}(i)[-\beta \tilde{P}_n(i) + A^T(i)\tilde{P}_n(i) + \tilde{P}_n(i)A(i) - P_0(i)B(i)R^{-1}(i)B^T(i)\tilde{P}_n(i) \\ &- \tilde{P}_n(i)B(i)R^{-1}(i)B^T(i)P_0(i) + \sum_{j \in S} \nu_{ij}\tilde{P}_n(j) - f_n(i)], \end{aligned}$$

Proof

1. In the proof of theorem 4.2, we showed that $P_0(i)$, $i \in S$, is given by (24) and (28) and this conclude (1).

2. μ admits the eigenvalue 0 because $\sum_{j \in S} \mu_{ij} = 0$. This eigenvalue is semi simple (the eigenspace associated to the eigenvalue 0 admits an eigenvectors basis) this can be shown easily by noting that $q_{\bar{\xi}}$, $\bar{\xi} \in \bar{S}$ defined before, form a basis of the c-dimensional subspace $Ker(\mu)$. From this property we have the decomposition: $\mathbb{R}^s = Ker(\mu) \oplus Im(\mu)$, where $Ker(\mu)$ denotes the kernel and $Im(\mu)$ the range of the operator μ. Therefore, $(P_n)_{kl} = (\bar{P}_n)_{kl} + (\tilde{P}_n)_{kl}$ where $(\bar{P}_n)_{kl} \in Ker(\mu)$, $(\tilde{P}_n)_{kl} \in Im(\mu)$, and $1 \leq k, l \leq p$. Substituting this structure for $n > 0$ in (7), we obtain that $(\tilde{P}_n)$ is solution (39) and since $\tilde{P}_n \in Im(\mu)$, then (39) admits a unique solution.

3. Since $(\bar{P}_n)_{kl} \in Ker(\mu)$, then

$$\begin{cases} \bar{P}_n(i) = \bar{P}_n(\bar{\xi}) & i \in S \\ \bar{P}_n(i) = \sum_{\bar{\xi}' \in \bar{S}} q_{\bar{\xi}'}(i)\bar{P}_n(\bar{\xi}'). \end{cases} \tag{41}$$

Substituting this structure (41) in (7), multiplying each equation by $m_{\bar{\xi}}$ such that $i \in \bar{\xi}$ and summing up over $\bar{\xi}$, we obtain:

$$\beta \bar{P}_n(\bar{\xi}) = D^T(\bar{\xi})\bar{P}_n(\bar{\xi}) + \bar{P}_n(\bar{\xi})D(\bar{\xi}) + \sum_{\bar{\xi}' \in \bar{S}} \nu_{\bar{\xi}\bar{\xi}'} \bar{P}_n(\bar{\xi}') + T_n(\bar{\xi}), \quad \bar{\xi} \in \bar{S}, \tag{42}$$

By using the same procedure to derive the uniqueness of equation (34), we conclude that equation (42) admits a unique solution. ∎

We present an iterative algorithm to compute P_0 and another algorithm to compute $\bar{P}_n$, $n > 0$.

Iterative algorithm to compute $\bar{P}_0$:

1- Set $\bar{P}_0^0(i) = 0$, $i \in S$.
2- Compute $\bar{P}_0^l(i), i \in S, l \geq 0$, iteratively by solving the following set of quadratic equations:

$$(\beta+\nu_{\bar{\xi}})\bar{P}_0^{l+1}(\bar{\xi}) = \bar{A}^T(\bar{\xi})\bar{P}_0^{l+1}(\bar{\xi})+\bar{P}_0^{l+1}(\bar{\xi})\bar{A}(\bar{\xi})-\bar{P}_0^{l+1}(\bar{\xi})\bar{B}(\bar{\xi})\bar{B}^T(\bar{\xi})\bar{P}_0^{l+1}(\bar{\xi})+\tilde{Q}(\bar{P}_0^l,\bar{\xi}), \tag{43}$$

where

$$\tilde{Q}(\bar{P}_0^l,\bar{\xi}) = \bar{Q}(\bar{\xi}) + \sum_{\bar{\xi}\in\bar{S}} \nu_{\bar{\xi}\bar{\xi}'}\bar{P}_0^l(\bar{\xi}) + \nu_{\bar{\xi}}\bar{P}_0^l(\bar{\xi})$$

and $\nu_{\bar{\xi}} =\mid \nu_{\bar{\xi}\bar{\xi}} \mid$
Note that the sequence thus generated is monotonically nondecreasing [15], and therefore possess limits, with former satisfying (38).

Iterative algorithm to compute $\bar{P}_n$, $n > 0$

Let H be a solution to:

$$\beta H(\bar{\xi}) = I + D^T(\bar{\xi})H(\bar{\xi}) + H(\bar{\xi})D(\bar{\xi}) + \sum_{\bar{\xi}\in\bar{S}} \nu_{\bar{\xi}\bar{\xi}'}H(\bar{\xi}'). \tag{44}$$

Choose ρ such that the matrix $\widehat{T}_n(\bar{\xi}) = T_n(\bar{\xi}) + \rho I$ is positive definite (where T_n is defined in (5.1)).
1- Set $\bar{P}_n^0(\bar{\xi}) := \rho H(\bar{\xi})\ \forall\bar{\xi} \in \bar{S}$.
2- Compute $\bar{P}_n^l(\bar{\xi})$, $\bar{\xi} \in \bar{S}$, $l \geq 1$, iteratively by solving the following set of coupled algebraic Lyapunov equations:

$$(\beta + \nu_{\bar{\xi}})\bar{P}_n^{l+1}(\bar{\xi}) = D^T(\bar{\xi})\bar{P}_n^{l+1}(\bar{\xi}) + \bar{P}_n^{l+1}(\bar{\xi})D(\bar{\xi}) + \bar{T}(\bar{P}_n^l(\bar{\xi}),\bar{\xi}), \tag{45}$$

where $\bar{T}(\bar{P}_n^l(\bar{\xi}),\bar{\xi}) = \sum_{\bar{\xi}'\in\bar{S}} \nu_{\bar{\xi}\bar{\xi}'}\bar{P}_n^l(\bar{\xi}') + T_n(\bar{\xi}) + \nu_{\bar{\xi}}\bar{P}_n^l(\bar{\xi})$.

Let $\widehat{P}_n(\bar{\xi}) = \bar{P}_n(\bar{\xi}) + \rho H(\bar{\xi})$, then from (42) we have

$$\begin{aligned}(\beta + \nu_{\bar{\xi}})\widehat{P}_n(\bar{\xi}) &= D^T(\bar{\xi})\widehat{P}_n(\bar{\xi}) + \widehat{P}_n(\bar{\xi})D(\bar{\xi}) + \sum_{\bar{\xi}' \in \bar{S}} \nu_{\bar{\xi}\bar{\xi}'}\widehat{P}_n(\bar{\xi}') + T_n(\bar{\xi}) + \nu_{\bar{\xi}}\widehat{P}_n(\bar{\xi}) \\ &+ \rho[(\beta + \nu_{\bar{\xi}})H(\bar{\xi}) - D^T(\bar{\xi})H(\bar{\xi}) - H(\bar{\xi})D(\bar{\xi}) - \sum_{\bar{\xi} \in \bar{S}} \nu_{\bar{\xi}\bar{\xi}'}H(\bar{\xi}') \\ &- \nu_{\bar{\xi}}H(\bar{\xi})]. \end{aligned} \tag{46}$$

By using the same procedure to derive equation (34), we conclude that equation (44) admits a unique positive solution (since $I \geq 0$). Hence the equation (46) becomes:

$$\begin{aligned}(\beta + \nu_{\bar{\xi}})\widehat{P}_n(\bar{\xi}) &= D^T(\bar{\xi})\widehat{P}_n(\bar{\xi}) + \widehat{P}_n(\bar{\xi})D(\bar{\xi}) + \sum_{\bar{\xi}' \in \bar{S}} \nu_{\bar{\xi}\bar{\xi}'}\widehat{P}_n(\bar{\xi}') \\ &+ \nu_{\bar{\xi}}\widehat{P}_n(\bar{\xi}) + T_n(\bar{\xi}) + \rho I \end{aligned} \tag{47}$$

The equation (47) admits a unique positive solution $\widehat{P}_n$ which can be computed by the following iterative algorithm
1- Set $\widehat{P}_n^0(\bar{\xi}) := 0$, $\bar{\xi} \in \bar{S}$.
2- Compute $\widehat{P}_n^l(\bar{\xi})$, $\bar{\xi} \in \bar{S}$, $l \geq 1$, iteratively by solving the following set of coupled algebraic Lyapunov equations:

$$(\beta + \nu_{\bar{\xi}})\widehat{P}_n^{l+1}(\bar{\xi}) = D^T(\bar{\xi})\widehat{P}_n^{l+1}(\bar{\xi}) + \widehat{P}_n^{l+1}(\bar{\xi})D(\bar{\xi}) + \tilde{T}_n(\widehat{P}_n^l, \bar{\xi}), \tag{48}$$

where

$$\tilde{T}_n(\widehat{P}_n^l, \bar{\xi}) = \widehat{T}(\bar{\xi}) + \sum_{\bar{\xi}' \in \bar{S}} \nu_{\bar{\xi}\bar{\xi}'}\widehat{P}_n^l(\bar{\xi}') + \nu_{\bar{\xi}}\widehat{P}_n^l(\bar{\xi}).$$

From the definition of $\tilde{T}_n(\widehat{P}_n^l, \bar{\xi})$,

$$\text{if} \quad \widehat{P}_n^{l+1}(\bar{\xi}) \geq \widehat{P}_n^l(\bar{\xi}) \quad \forall \bar{\xi} \in \bar{S} \quad \text{then} \quad \tilde{T}_n(\widehat{P}_n^{l+1}, \bar{\xi}) \geq \tilde{T}_n(\widehat{P}_n^l, \bar{\xi}) \quad \forall \bar{\xi} \in \bar{S}. \tag{49}$$

Note that if $\widehat{P}_n^{l+2}$ and $\widehat{P}_n^{l+1}$ are the solutions to

$$(\beta + \nu_{\bar{\xi}})\widehat{P}_n^{l+2}(\bar{\xi}) = D^T(\bar{\xi})\widehat{P}_n^{l+2}(\bar{\xi}) + \widehat{P}_n^{l+2}(\bar{\xi})D(\bar{\xi}) + \tilde{T}_n(\widehat{P}_n^{l+1}, \bar{\xi})$$

and

$$\beta + \nu_{\bar{\xi}})\widehat{P}_n^{l+1}(\bar{\xi}) = D^T(\bar{\xi})\widehat{P}_n^{l+1}(\bar{\xi}) + \widehat{P}_n^{l+1}(\bar{\xi})D(\bar{\xi}) + \tilde{T}_n(\widehat{P}_n^l, \bar{\xi})$$

respectively, then by subtraction we obtain:

$$\begin{aligned}(\beta + \nu_{\bar{\xi}})(\widehat{P}_n^{l+2}(\bar{\xi}) - \widehat{P}_n^{l+1}(\bar{\xi})) &= D^T(\bar{\xi})(\widehat{P}_n^{l+2}(\bar{\xi}) - \widehat{P}_n^{l+1}(\bar{\xi})) + (\widehat{P}_n^{l+2}(\bar{\xi}) - \widehat{P}_n^{l+1}(\bar{\xi}))D(\bar{\xi}) \\ &+ (\tilde{T}_n(\widehat{P}_n^{l+1}, \bar{\xi}) - \tilde{T}_n(\widehat{P}_n^l, \bar{\xi})) \end{aligned} \tag{50}$$

If

$$\tilde{T}_n(\widehat{P}_n^{l+1}, \bar{\xi}) \geq \tilde{T}_n(\widehat{P}_n^{l}(\bar{\xi}), \bar{\xi}),$$

then the last equation admits a unique positive solution, hence

$$\widehat{P}_n^{l+2}(\bar{\xi}) \geq \widehat{P}_n^{l+1}(\bar{\xi}). \tag{51}$$

In view of the fact that $\widehat{P}_n^1(\bar{\xi}) \geq \widehat{P}_n^0(\bar{\xi}) := 0$, (49) (51) establish by induction the desired result that the sequences $\widehat{P}_n^l$ are monotonically nondecreasing sequences, which have to converge to the solution of (47). It can be also observed that the sequences $\bar{P}_n^l$ defined above satisfy $\bar{P}_n^l = \widehat{P}_n^l - \rho H$, hence the sequences $\bar{P}_n^l$ are monotonically nondecreasing sequences which have to converge to the solution of (42).

Remark 3 *The system can be solved recursively in n:*
P_0 *is computed by the iterative algorithm,*
$\tilde{P}_1$ *is solution of* $(39)_1$*,*
$\bar{P}_1$ *is computed by the iterative algorithm ,*
$\tilde{P}_2$ *is solution of* $(39)_2$*,*
$\bar{P}_2$ *is computed by the iterative algorithm ,*
and so on .....

6 Conclusion

In this paper, we have presented a study to the perturbation of multidimensional linear quadratic systems with Markov jump parameters. Under perfect state measurements for infinite horizon, we have showed that the optimal policy and value admit a Taylor expansion in the perturbation. This was done by using the solution of a corresponding set of CARE (coupled algebraic Riccati equations). A computation of the Taylor expansion was presented.

Results presented here could be extended in several direction. The first, is the case where the Markov chain is controlled too. In [8] we already investigated that direction, however we restricted to the case in which the state is one dimensional; the quadratic structure of the optimal value does not seem to extend to multidimensional case. We shall carry on future research to study the limit of the optimal value for such hybrid systems, as the singular perturbation parameter tends to zero. Other extensions are to problems in which the linear dynamics has another component of noize, in which case, one can still obtain the linear quadratic structure for the Gaussian disturbance, or for the worst case analysis (formulated as an H^∞ control problem).

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